Influences of constraint errors on the mobility of a 3-DOF translational parallel manipulator

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Abstract—The influences of geometric constraint errors on the mobility of a 3-DOF translational parallel manipulator (TPM) are discussed in this paper. It is demonstrated based on screw theory that the mobility of the TPM may vanish if there exist manufacturing and installation errors violating the constraint conditions. To avoid such a zero-mobility case, the original manipulator is converted into a non-overconstrained one by adding a revolute joint in each limb. It is shown that the conversion brings no effects to the theoretical mobility and kinematics of the original TPM, but it will take effect in case of constraint errors. Moreover, the influences of geometric constraint errors on the mobility of a converted manipulator are investigated, which provides an insight into the effects of constraint errors.

Keywords: parallel manipulator, constraint error, mobility

I. Introduction

With a considerate design, parallel manipulators can provide several attractive advantages over their serial counterparts in terms of high stiffness, high accuracy, and low inertia, since the performances of parallel manipulators are very sensitive to their geometry [1]. In other words, a parallel manipulator may not exhibit the expected merits if it is not carefully designed. Particularly, this is true if the accuracy property of a parallel manipulator is taken into account. And this is the reason why many efforts have been paid towards the optimal design of a parallel manipulator for best accuracy performance in the literatures [2, 3].

In addition to the architecture design, the accuracy of a parallel manipulator relies on many other factors including the actuator control errors, installation errors, manufacturing tolerances and clearances, etc. As is well known, there are a class of lower-mobility parallel manipulators with less than six DOF (degree of freedom), whose mobility is guaranteed by some certain geometrical constraints. For instance, a spherical 3-DOF parallel manipulator [4] demands that all of the revolute joint axes intersect at one common point, and a translational 3-UPU (three-universal-prismatic-universal) parallel mechanism [5] requires that the joint axis of two outer and two inner revolute (R) joints are parallel to each other, respectively. It is reported that the 3-UPU platform is highly sensitive to certain minute clearances in the U joints [6]. And recently, the effects of geometrical constraint errors on a 6-DOF decoupled parallel manipulator are presented in [7] via the screw theoretical approach. However, there are seldom literatures focusing on the investigation of the influences of geometric constraint errors on overconstrained parallel manipulators.

In this paper, we will analyze the influences of geometrical constraint errors on the mobility of an overconstrained 3-PRC (three-prismatic-revolute-cylindrical) TPM [8] by resorting to screw theory. It will be shown that the mobility of the manipulator vanishes if there exist some errors violating the geometric constraint conditions. Under such situation, the accuracy analysis of a 3-PRC TPM is meaningless. Motivated by this point, the original 3-PRC TPM is converted into a non-overconstrained manipulator without influencing its theoretical mobility and kinematics. Furthermore, the effects of constraint errors on the mobility of the resulted manipulator are investigated in details.

II. Architecture description of the manipulator

The schematic diagram of a 3-PRC TPM is shown in Fig. 1. It consists of a mobile platform, a fixed base, and three limbs with identical kinematic structure. Each limb connects the fixed base to the mobile platform by a P (prismatic) joint, a R (revolute) joint, and a C (cylindrical) joint in sequence, where the P joint is driven by a linear actuator...
mounted on the fixed base.

To facilitate the analysis, as shown in Figs. 1 and 2, we assign a fixed Cartesian frame \( O(x, y, z) \) at the centered point \( O \) of the fixed base, and a moving frame \( P(u, v, w) \) on the triangle mobile platform at the centered point \( P \), along with the \( x- \) and \( y- \) axes parallel to the \( u- \) and \( v- \) axes, respectively.

In addition, the \( i \)-th limb \( C_iB_i \) (\( i = 1, 2, 3 \)) with the same length \( l \) is connected to the mobile platform at \( B_i \) which is a point on the axis of the \( i \)-th C joint. \( B'_i \) denotes the point on the mobile platform being coincident with the initial position of \( B_i \), and the three points \( B'_i \), for \( i = 1, 2, 3 \), lie on a circle of radius \( b \). The three rails \( M_iN_i \) intersect one another at point \( D \) and intersect the \( x- \) \( y- \) plane at points \( A_1 \), \( A_2 \), and \( A_3 \) that lie on a circle of radius \( \alpha \). Moreover, the axis of P joint is perpendicular to the axes of R and C joints within the \( i \)-th limb. Angle \( \alpha \) is measured from the fixed base to rails \( M_iN_i \) and is defined as the layout angle of actuators. In order to obtain a compact architecture, the value of \( \alpha \) is designed within the range of \( 0^\circ \leq \alpha \leq 90^\circ \). Angle \( \varphi_j \) is defined from the \( x- \) axis to \( OA_i \) in the fixed frame, and also from the \( u- \) axis to \( PB'_i \) in the moving frame. Without loss of generality, let the \( x- \) axis point along \( OA_i \), and the \( u- \) axis along \( PB'_i \). Then, we have \( \varphi_1 = 0^\circ \).

Furthermore, in order to achieve a symmetric workspace of the manipulator, both \( \Delta A_1A_2A_3 \) and \( \Delta B_1B_2B_3 \) are assigned to be equilateral triangles. It has been shown [8] that in order to prevent the mobile platform from changing its orientation, it is sufficient for the joint axes within the same limb to satisfy some geometric constraint conditions, i.e., the R joint axis \( (s_{2,i}) \) and C joint axis \( (s_{3,i}) \) or \( s_{4,i} \) within the \( i \)-th limb are parallel to the same unit vector \( s_{0,i} \), for \( i = 1, 2, 3 \).

III. Mobility of a 3-PRC TPM

The mobility determination, i.e., the DOF identification, is the first and foremost issue in designing a parallel manipulator. The number of DOF of a 3-PRC TPM given by the general Grübler-Kutzbach criterion is

\[
F = \lambda(n - j - 1) + \sum_{i=1}^{j} f_i = 6 \times (8 - 9 - 1) + 12 = 0 \quad (1)
\]

where \( \lambda \) represents the dimension of task space, \( n \) is the number of links, \( j \) is the number of joints, and \( f_i \) denotes the degrees of freedom of joint \( i \).

Equation (1) indicates that a 3-PRC TPM employs an overconstrained mechanism. With the aforementioned constraint geometric conditions satisfied, it has been mathematically shown that the 3-PRC parallel manipulator possesses three spatial translational DOF [8].

IV. The effect of constraint errors on a 3-PRC TPM

In practice, due to the unavoidable errors subject to manufacturing tolerance, imperfect installation, etc., the axis of R joint may not parallel to the C joint axis in the same limb, i.e., \( s_{2,i} \parallel s_{3,i} \), for \( i = 1, 2, 3 \).

With \( \omega = [\omega_x \ \omega_y \ \omega_z]^T \) and \( \nu = [v_x \ v_y \ v_z]^T \) respectively denoting vectors for the angular and linear velocities, the twist of the mobile platform can be defined as \( \mathbf{s}_p = [\omega^T \ \nu^T]^T \). Considering that a C joint is equivalent to the combination of a P joint with a coaxial R joint (\( s_{3,i} = s_{4,i} = s_{0,i} \)), the connectivity of each limb for a 3-PRC TPM is equal to 4 since each limb consists of four 1-DOF joints. Hence, the instantaneous twist \( \mathbf{s}_p \) of the mobile platform can be expressed as a linear combination of the 4 instantaneous twists, i.e.,

\[
\mathbf{s}_p = d_i\hat{s}_{1,i} + \hat{\theta}_{2,i}\hat{s}_{2,i} + \hat{\theta}_{3,i}\hat{s}_{3,i} + \hat{s}_{4,i}\hat{s}_{4,i} \quad (2)
\]

for \( i = 1, 2, 3 \), where \( \hat{\theta}_{j,i} \) is the intensity and \( \hat{s}_{j,i} \) denotes a unit vector associated with the \( j \)-th joint of the \( i \)-th limb with respect to the instantaneous reference frame \( P_i \) and

\[
\hat{s}_{1,i} = \begin{bmatrix} 0 \\ s_{1,i} \end{bmatrix} \quad (3)
\]
\[
\hat{s}_{2,i} = \begin{bmatrix} s_{2,i} \\ c_i \times s_{2,i} \end{bmatrix} \quad (4)
\]
\[
\hat{s}_{3,i} = \begin{bmatrix} s_{3,i} \\ b_i \times s_{3,i} \end{bmatrix} \quad (5)
\]
\[
\hat{s}_{4,i} = \begin{bmatrix} 0 \\ s_{4,i} \end{bmatrix} \quad (6)
\]

can be identified, where \( s_{j,i} \) represents a unit vector along the \( j \)-th joint axis of the \( i \)-th limb, \( \mathbf{0} \) denotes a \( 3 \times 1 \) zero vector, \( \mathbf{b}_i = PB_i \), and \( \mathbf{c}_i = PC_i = \mathbf{b}_i - l_1\mathbf{0} \).

The screws that are reciprocal to all the joint screws of one limb of a 3-PRC TPM form a 2-system. Assume the reciprocal screws have the following form:

\[
\mathbf{s}_{r,k,i} = \begin{bmatrix} s_{r,k,i} \\ h_{k,i} \end{bmatrix} \quad (7)
\]
where $s_{r,k,i} = \begin{bmatrix} s_{r,k,i,1} & s_{r,k,i,2} & s_{r,k,i,3} \end{bmatrix}^T$ and $h_{k,i} = \begin{bmatrix} h_{k,i,1} & h_{k,i,2} & h_{k,i,3} \end{bmatrix}^T$, for $k = 1, 2$, and $i = 1, 2, 3.$

According to the definition of reciprocal screws [9], the following relationships can be derived:

$$s_{r,k,i} \cdot s_{1,i} = 0$$  \hspace{1cm} (8)

$$s_{r,k,i} \cdot (c_i \times s_{2,i}) + h_{k,i} \cdot s_{2,i} = 0$$  \hspace{1cm} (9)

$$s_{r,k,i} \cdot (b_i \times s_{3,i}) + h_{k,i} \cdot s_{3,i} = 0$$  \hspace{1cm} (10)

$$s_{r,k,i} \cdot s_{1,i} = 0.$$  \hspace{1cm} (11)

Considering Eqs. (8) and (11) simultaneously, we can intuitively obtain the expressions for $s_{r,k,i}$ ($k = 1$ and 2), i.e.,

$$s_{r,1,i} = 0$$

$$s_{r,2,i} = s_{1,i} \times s_{4,i}.$$  \hspace{1cm} (13)

Then, substituting $s_{r,k,i}$ into Eqs. (9) and (10), allows the generation of $h_{k,i}$ for $k = 1$ and 2, respectively:

$$h_{1,i} = s_{2,i} \times s_{3,i}$$  \hspace{1cm} (14)

$$h_{2,i} = [h_{2,i,1} \ a_{2,i,2} \ a_{2,i,3}]^T$$  \hspace{1cm} (15)

where the expressions of $h_{2,i,1}, h_{2,i,2}$, and $h_{2,i,3}$ are omitted here. Thus, the two reciprocal screws $s_{r,k,i}$ ($k = 1$ and 2) for each limb are all identified.

Taking the reciprocal product of both sides of Eq. (2) with $s_{r,1,i}$, yields

$$\begin{bmatrix} h_{2,i,1}^T & 0^T \\ h_{2,i,2}^T & 0^T \\ h_{2,i,3}^T & 0^T \end{bmatrix} \begin{bmatrix} \omega \\ v \end{bmatrix} = 0.$$  \hspace{1cm} (16)

If $h_{2,i,i}$, for $i = 1, 2$ and 3, are linearly independent, the unique solution to Eq. (16) is

$$\omega = 0.$$  \hspace{1cm} (17)

Similarly, taking the reciprocal product of Eq. (2) with $s_{r,2,i}$, yields

$$\begin{bmatrix} h_{2,i,1}^T & s_{r,2,1}^T \\ h_{2,i,2}^T & s_{r,2,2}^T \\ h_{2,i,3}^T & s_{r,2,3}^T \end{bmatrix} \begin{bmatrix} \omega \\ v \end{bmatrix} = 0.$$  \hspace{1cm} (18)

When the layout angle of actuators $0^\circ < \alpha < 90^\circ$, the three vectors of $s_{r,2,i}$ ($i = 1, 2, 3$) are always linear independent. Then, in view of Eqs. (17) and (18), we can derive that

$$\omega = 0.$$  \hspace{1cm} (19)

Equations (17) and (19) indicate that, if the constraint conditions of a 3-PRC TPM are violated due to the constraint errors in the R and C joint axes, the mobility of the manipulator vanishes. In other words, when the axes of the R and C joints within the same limb are nonparallel, that most likely occurs in practice due to the unavoidable manufacturing tolerance and installation errors, the mobile platform of the manipulator will not move at all even if driven by the actuators. It is predicted that the movable problem can be solved by converting the original 3-PRC TPM into a non-overconstrained manipulator [8], which will be implemented in the following section.

V. Elimination of overconstraints of a 3-PRC TPM

Motivated by the above discussions, a R joint is added in each limb of the 3-PRC TPM to eliminate the overconstrained conditions. By resorting to screw theory, it will be demonstrated that the added R joints bring no effects to the kinematics and mobility of an ideal 3-PRC TPM, since there are inactive when there are no constraint errors. Whereas they are introduced to eliminate the overconstraints of the 3-PRC TPM, and they will take effect in the presence of constraint errors.

For the convenience of practical assembly, the R joint is added with the common axis of the P joint in each limb, which allows the generation of a 3-CRC parallel manipulator indeed. The joint screw of the added R joint can be expressed as:

$$\hat{s}_{5,i} = \begin{bmatrix} s_{5,i} \\ d_i \times s_{5,i} \end{bmatrix}$$  \hspace{1cm} (20)

where $s_{5,i} = s_{1,i}$ represents a unit vector along the actuated P joint and the added R joint as well, and $d_i = c_i$ is the position vector of the added R joint with respect to the reference frame $P$.

Then, the instantaneous twist $\hat{s}_p$ of the mobile platform can be expressed as a linear combination of the 3 instantaneous twists, i.e.,

$$\hat{s}_p = \dot{d}_i \hat{s}_{1,i} + \dot{\theta}_{2,i} \hat{s}_{2,i} + \dot{\theta}_{3,i} \hat{s}_{3,i} + \dot{s}_4 \hat{s}_{4,i} + \dot{\theta}_{5,i} \hat{s}_{5,i}$$  \hspace{1cm} (21)

for $i = 1, 2$, and 3.

The screws that are reciprocal to all the joint screws of a limb of the manipulator form a 1-system. Hence, one unit reciprocal screw of the $i$-th limb can be identified as an infinite-pitch wrench screw as follows.

$$\hat{s}_{r,i} = \begin{bmatrix} 0 \\ k_i \end{bmatrix}$$  \hspace{1cm} (22)

where $k_i$ (Fig. 2) is a unit vector defined by

$$k_i = \frac{s_{2,i} \times s_{5,i}}{|s_{2,i} \times s_{5,i}|}.$$  \hspace{1cm} (23)

Taking the reciprocal product of both sides of Eq. (21) with $\hat{s}_{r,i}$, yields

$$\begin{bmatrix} k_{1,i}^T & 0^T \\ k_{2,i}^T & 0^T \\ k_{3,i}^T & 0^T \end{bmatrix} \begin{bmatrix} \omega \\ v \end{bmatrix} = 0.$$  \hspace{1cm} (24)

It is observed that each row of the first matrix in Eq. (24) represents a unit couple of constraints imposed by the joints
of the \( i \)-th limb, and the couple is exerted on the mobile platform around the direction of \( \mathbf{k}_i \). When the layout angles of actuators are restricted within the range of \( 0^\circ < \alpha < 90^\circ \), the three vectors of \( \mathbf{k}_i \), for \( i = 1, 2, \) and 3, are always linear independent. Hence, the combination of the couples constrains the mobile platform to a 3-DOF motion. Moreover, the unique solution to Eq. (24) is

\[
\omega = \mathbf{0} \quad (25)
\]

which exhibits that the 3-CRC manipulator has three translational DOF as well.

Substituting Eq. (25) into Eq. (21), results in

\[
\dot{\theta}_{2,i} \mathbf{s}_{2,i} + \dot{\theta}_{3,i} \mathbf{s}_{3,i} + \dot{\theta}_{5,i} \mathbf{s}_{5,i} = \mathbf{0}. \quad (26)
\]

In view of the condition that \( \mathbf{s}_{2,i} = \mathbf{s}_{3,i} \neq \mathbf{s}_{5,i} \), the following two equations can be derived from Eq. (25):

\[
\dot{\theta}_{2,i} + \dot{\theta}_{3,i} = 0 \quad (27)
\]

\[
\dot{\theta}_{5,i} = 0. \quad (28)
\]

Additionally, the DOF number of a 3-CRC parallel manipulator can be given by the general Grubler-Kutzbach criterion as:

\[
F = \lambda(n-j-1) + \sum_{i=1}^{j} f_i = 6 \times (8-9-1) + 15 = 3 \quad (29)
\]

which indicates that it is not overconstrained.

Equation (28) reveals that the added R joints in a 3-CRC TPM do not rotate around the C joints at all and do not influence the mobility and kinematics of the original 3-PRC TPM in ideal conditions, but they are introduced to eliminate the internal constraints in a 3-PRC TPM and lead to a non-overconstrained manipulator as indicated by Eq. (29).

It is noticeable that the overconstrains of a 3-PRC TPM can be eliminated by other approaches as well. For example, one R joint can be added between the original P and R joints to construct a 3-PUC parallel manipulator, which is also a non-overconstrained manipulator. And it can be shown easily via screw theory that the 3-PUC parallel manipulator also has the same mobility and kinematics with the original 3-PRC TPM.

VI. Constraint errors analysis of a 3-CRC TPM

The 3-CRC TPM is designed to meet the constraint conditions addressed at the end of Section III. In addition, for the convenience of assembly, the leg \( C_iB_i \) is designed to be perpendicular to the axes of the R and C joints mounted at its two end-points. The errors arising from the manufacturing tolerance and imperfect installation in these two kinds of geometric conditions will be considered for a 3-CRC TPM.

A. Case I: Nonparallelism error

First, let’s consider the case when the R and C joint axes are nonparallel, while they are still perpendicular to the leg direction, i.e., \( \mathbf{s}_{2,i} \parallel \mathbf{s}_{3,i}, \mathbf{s}_{2,i} \perp \mathbf{l}_{00} \), and \( \mathbf{s}_{3,i} \perp \mathbf{l}_{00} \), for \( i = 1, 2, \) and 3.

Under this case, we assume that one unit screw \( \hat{\mathbf{s}}_{c,i} \) being reciprocal to the five joint screws of the \( i \)-th limb takes the following form:

\[
\hat{\mathbf{s}}_{c,i} = \begin{bmatrix} \mathbf{s}_{r,i} \\ \mathbf{h}_{i} \end{bmatrix} \quad (30)
\]

where \( \mathbf{s}_{r,i} = \begin{bmatrix} r_{r,i1} & r_{r,i2} & r_{r,i3} \end{bmatrix}^T \) and \( \mathbf{h}_{i} = [h_{i1} \ h_{i2} \ h_{i3}]^T \), for \( i = 1, 2, \) and 3.

According to the definition of reciprocal screws [9], in view of Eqs. (3)—(6), (20) and (30), the following equations can be obtained:

\[
\mathbf{s}_{r,i} \cdot \mathbf{s}_{1,i} = 0 \quad (31)
\]

\[
\mathbf{s}_{r,i} \cdot (\mathbf{c}_i \times \mathbf{s}_{2,i}) + \mathbf{h}_i \cdot \mathbf{s}_{2,i} = 0 \quad (32)
\]

\[
\mathbf{s}_{r,i} \cdot (\mathbf{b}_i \times \mathbf{s}_{3,i}) + \mathbf{h}_i \cdot \mathbf{s}_{3,i} = 0 \quad (33)
\]

\[
\mathbf{s}_{r,i} \cdot \mathbf{s}_{4,i} = 0 \quad (34)
\]

\[
\mathbf{s}_{r,i} \cdot (\mathbf{c}_i \times \mathbf{s}_{5,i}) + \mathbf{h}_i \cdot \mathbf{s}_{5,i} = 0. \quad (35)
\]

From Eqs. (31) and (34), we can get that

\[
\mathbf{s}_{r,i} = 0 \quad \text{or} \quad \mathbf{s}_{r,i} = \frac{\mathbf{s}_{1,i} \times \mathbf{s}_{4,i}}{||\mathbf{s}_{1,i} \times \mathbf{s}_{4,i}||}. \quad (36)
\]

Substituting \( \mathbf{s}_{r,i} = 0 \) into Eqs. (32), (33) and (35), we can derive that \( \mathbf{h}_i = \mathbf{0} \) (discarded) since the three vectors \( \mathbf{s}_{2,i}, \mathbf{s}_{3,i}, \) and \( \mathbf{s}_{5,i} \) are neither coplanar nor parallel to one another. In the same way, with the substitution of the second value of \( \mathbf{s}_{r,i} \), we can derive that

\[
\mathbf{h}_i = -\begin{bmatrix} \mathbf{s}_{2,i} \\ \mathbf{s}_{3,i} \\ \mathbf{s}_{5,i} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{s}_{r,i} \cdot (\mathbf{c}_i \times \mathbf{s}_{2,i}) \\ \mathbf{s}_{r,i} \cdot (\mathbf{b}_i \times \mathbf{s}_{3,i}) \\ \mathbf{s}_{r,i} \cdot (\mathbf{c}_i \times \mathbf{s}_{5,i}) \end{bmatrix}. \quad (37)
\]

Taking the reciprocal product of both sides of Eq. (21) with \( \hat{\mathbf{s}}_{c,i} \), yields

\[
\mathbf{J}_c^r \mathbf{s}_p = \mathbf{0} \quad (38)
\]

where

\[
\mathbf{J}_c^r = \begin{bmatrix} \mathbf{h}_i^T & \mathbf{s}_{r,i}^T & \mathbf{h}_i^T \end{bmatrix} = \begin{bmatrix} \mathbf{J}_c^r & \mathbf{J}_c^r \end{bmatrix} \quad (39)
\]

is called the Jacobian of constraints [10]. Each row in \( \mathbf{J}_c^r \) represents a constraint force exerted on the mobile platform of the manipulator. The combination of the forces constrains the manipulator to a 3-DOF motion. In view of Eq. (38), it can be observed that the 3-CRC parallel manipulator can not provide a pure translation motion, and the rotation and translation of the manipulator is coupled by the matrix \( \mathbf{J}_c^r \) and can be expressed as follows:

\[
\mathbf{J}_c^r \mathbf{\omega} + \mathbf{J}_c^r \mathbf{\upsilon} = \mathbf{0}. \quad (40)
\]
Let the P joint in each limb be locked, then the reciprocal screws of each limb form a 2-system which includes one screw \( \mathbf{s}_{c,i} \) identified in Eq. (30). The other basis screw being reciprocal to all the passive joint screws of the \( i \)-th limb can be identified as a zero-pitch screw along the direction of the \( i \)-th leg, i.e.,

\[
\hat{s}_{0,i} = \begin{bmatrix} l_{0i} \end{bmatrix}
\]  

(41)

where \( l_{0i} \) is a unit vector along \( \overrightarrow{CiB_i} \).

Taking the reciprocal product of both sides of Eq. (21) with \( \hat{s}_{0,i} \), results in

\[
\mathbf{J}_a \hat{s}_p = \dot{\mathbf{q}}
\]

(42)

where \( \dot{\mathbf{q}} = [\dot{d}_1 \; \dot{d}_2 \; \dot{d}_3]^T \) denotes the actuated joint rates and

\[
\mathbf{J}_a = \begin{bmatrix}
\mathbf{J}_{01} & \mathbf{J}_{02} & \mathbf{J}_{03}
\end{bmatrix}
\]

(43)

is called the Jacobian of actuations [10], which transforms the angular and linear velocities of the mobile platform to the input joint rates.

**B. Case II: Nonperpendicularity error**

Second, let’s focus on the situation in case of the leg direction is not perpendicular to the R and C joint axes, whereas the R and C joints keep parallel to each other. Then, one unit screw being reciprocal to all the joint screws of the \( i \)-th limb of the manipulator forms a 1-system, which can be identified as follows.

\[
\hat{s}_{c,i} = \begin{bmatrix} 0 \\
\mathbf{k}_i \end{bmatrix}
\]

(44)

where \( \mathbf{k}_i \) is expressed in Eq. (23).

Taking the reciprocal product of both sides of Eq. (21) with \( \hat{s}_{c,i} \), yields Eq. (24), which can be written into:

\[
\mathbf{J}_a \hat{s}_p = 0
\]

(45)

where

\[
\mathbf{J}_a = \begin{bmatrix}
\mathbf{k}_1^T & 0^T \\
\mathbf{k}_2^T & 0^T \\
\mathbf{k}_3^T & 0^T
\end{bmatrix}
\]

(46)

From Eq. (45), it is observed that the 3-CRC parallel manipulator still possesses three translational DOF under such case.

Likewise, let the P joint in each limb be locked, then the reciprocal screws of each limb form a 2-system which includes one screw \( \mathbf{s}_{n,i} \) described in Eq. (44). The other one can be identified as a zero-pitch screw along the direction of \( \mathbf{f}_{0i} \), i.e.,

\[
\hat{s}_{n,i} = \begin{bmatrix} \mathbf{f}_{0i} \\
\mathbf{c}_i \times \mathbf{f}_{0i} \end{bmatrix}
\]

(47)

where \( \mathbf{f}_{0i} \) (Fig. 3) is a unit vector passing through point \( C_i \) and being perpendicular to the \( C \) joint axis direction in the \( i \)-th limb, for \( i = 1, 2, \) and 3.

Taking the reciprocal product of both sides of Eq. (21) with \( \hat{s}_{n,i} \), results in

\[
\mathbf{J}_a \hat{s}_p = \dot{\mathbf{q}}
\]

(48)

where the Jacobian of actuations becomes

\[
\mathbf{J}_n = \begin{bmatrix}
\mathbf{J}_{01} & \mathbf{J}_{02} & \mathbf{J}_{03}
\end{bmatrix}
\]

(49)

**C. Case III: Nonparallelism and nonperpendicularity errors**

It is observed that the nonparallelism error influences the rotation and translation of the manipulator via \( \mathbf{J}_n \) in Eq. (39) and the nonperpendicularity error only affects the translation motion reflected by \( \mathbf{J}_a \) in Eq. (49). Therefore, the effects of any overall constraint errors can be assessed by the superposition of the above two kinds of errors, i.e., the combination of Eqs. (39) and (49), which leads to

\[
\dot{\mathbf{q}}_o = \mathbf{J}_a \hat{s}_p
\]

(50)

where \( \dot{\mathbf{q}}_o = [\dot{d}_1 \; \dot{d}_2 \; \dot{d}_3 \; \dot{d}_4 \; \dot{d}_5 \; \dot{d}_6]^T \) is the vector of extended input joint rates, and

\[
\mathbf{J}_e = \begin{bmatrix}
\mathbf{J}_a \\
\mathbf{J}_n
\end{bmatrix} = \begin{bmatrix}
\mathbf{J}_{01} & \mathbf{J}_{02} & \mathbf{J}_{03} \\
\mathbf{J}_{10} & \mathbf{J}_{12} & \mathbf{J}_{13} \\
\mathbf{J}_{20} & \mathbf{J}_{22} & \mathbf{J}_{23}
\end{bmatrix}
\]

(51)

is called the overall Jacobian of a 3-CRC TPM, which includes the effects of constraint errors.
<table>
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<th>Parameter</th>
<th>$a$</th>
<th>$b$</th>
<th>$l$</th>
<th>$\alpha$</th>
<th>$\varphi_1$</th>
<th>$\varphi_2$</th>
<th>$\varphi_3$</th>
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<td>30 cm</td>
<td>50 cm</td>
<td>45°</td>
<td>0°</td>
<td>60°</td>
<td>120°</td>
</tr>
</tbody>
</table>

**TABLE I. Kinematic parameters of a 3-CRC TPM.**

In addition, combining Eqs. (43) with (46) allows the generation of the overall Jacobian matrix without the influences of constraint errors:

$$J = \begin{bmatrix} J_n \\ J_c \end{bmatrix} = \begin{bmatrix} \delta p_x \\ \delta p_y \\ \delta p_z \end{bmatrix}^T = \begin{bmatrix} \delta \theta_x \\ \delta \theta_y \\ \delta \theta_z \end{bmatrix}^T$$

which relates the twist of the mobile platform to $\delta q_o$ via

$$\delta q_o = J^T p. \quad (53)$$

Let $\delta p = [\delta p_x \ \delta p_y \ \delta p_z]^T$ and $\delta \theta = [\delta \theta_x \ \delta \theta_y \ \delta \theta_z]^T$ denote vectors of small translation and rotation around the reference frame, respectively. Then, the twist of the mobile platform can be expressed by $\delta q_p = [\delta \theta]^T \ [\delta p]^T$. Assume that there are no singularities, in case of a small displacement of each actuated joint $\delta q_i = [\delta d_1 \ \delta d_2 \ \delta d_3 \ 0 \ 0 \ 0]^T$, the twist can be calculated via Eqs. (50) and (55):

$$\delta s_p = (J^T)^{-1} \delta q_o, \ \delta s_p = (J)^{-1} \delta q_p \quad (54)$$

which represents the twist with and without the consideration of constraint errors, respectively.

### D. A case study

Let’s consider a 3-CRC TPM with the kinematic parameters shown in Table I, and the random constraint errors given in the following forms:

$$\delta s_{2,i} = [-0.1 \ 0.2 \ -0.3]^T, \ \delta s_{3,i} = [0.1 \ -0.1 \ 0.2]^T, \ \delta l_{10} = [0.2 \ -0.3 \ 0.4]^T. \quad (55)$$

The actual directions for the R and C joint axes and the legs are determined by adding the constraint errors to the corresponding nominal directions.

As for a small derivation of the actuated values $\delta q_o = [0.5 \ -0.6 \ 0.8 \ 0 \ 0 \ 0]^T$, the small twist of the mobile platform with and without the constraint errors can be respectively calculated as:

$$\delta s_p^c = [0.0014 \ -0.0019 \ 0.0035 \ 0.2074 \ 0.1063 \ -0.7493]^T, \ \delta s_p = [0 \ 0 \ 0 \ 0.2200 \ 0.1905 \ -0.7837]^T$$

where the first three elements denote the orientational errors in unit of radian, and the last three ones represent the positional errors in unit of centimeter.

It is seen that the 3-CRC manipulator possesses rotation motion which is contributed by the nonparallelism constraint errors, and translation motion which is influenced by both the nonparallelism and the nonperpendicularity errors.

### VII. Conclusions

In this paper, the constraint geometrical errors analysis for a 3-DOF parallel manipulator is carried out. It is shown that a 3-PRC TPM may not move in the presence of constraint errors due to the overconstraints. Thus, a 3-CRC parallel manipulator is proposed by adding a R joint in each limb of a 3-PRC TPM. The added R joints are shown to be inactive in the absence of constraint errors, but they will work if constraint errors exist. In addition, geometric errors are considered to have a view of constraint error effects on the 3-CRC TPM. It is observed that the nonparallelism errors existing between the passive R and C joints induce orientation errors to the mobile platform, which results in a 3-CRC manipulator with coupled motion of translation and rotation. Whereas the nonperpendicularity errors between the leg and C joint axis directions only affect the translational motion of the 3-CRC TPM.

This paper is only an attempt to solve the overconstraint problem of a 3-PRC TPM, which will provide a sound base for the accuracy analysis of a 3-CRC TPM considering both volumetric and constraint errors. Moreover, the methodology can be extended to constraint error analysis of other types of lower-mobility parallel manipulators as well.

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